Module-2 : properties of the Integers

Introduction

- Let the set 'Z' represents the set of all integers (Both positive and negative). A subset of this representing the set of all positive integers denoted by Z^+ play a significant role in establishing certain results/theorems.
- Given two distinct integers x, y they satisfy either of the two inequalities, $x > y$ or $y > x$. Therefore, we define the set Z^+ as follows

$$Z^+ = \{x / x \in Z, x > 0\} = \{x / x \in Z, x \geq 1\}$$

- Every non empty subset of Z^+ say Z_0^+ contains an integer x_0 such that $x_0 \leq x$ for all $x \in Z_0^+$. That is to say that set Z_0^+ contains a least/smallest element.

(*) Mathematical Induction

- The method of mathematical induction is based on a principle called the Induction principle. This principle can be proved by using another principle known as the Well-ordering principle. These two principles highlight some important properties of Integers.

(*) Well-ordering principle

- The Well-ordering principle states as follows:
- Every nonempty subset of Z^+ contains a smallest (least) element. The set Z^+ is well ordered.

(*) Induction principle (The principles of Mathematical Induction)

- The Induction principle states as follows:
- Let $S(n)$ denote an open statement that involves a positive integer 'n'. Suppose that the following conditions hold:

  (1) $S(1)$ is true or $S(n)$ is true for $n=1$.

  (2) If whenever $S(K)$ is true for some arbitrary element say $K \in Z^+$, then $S(K+1)$ is true

- Then $S(n)$ is true for all $n \in Z^+$.

(*) Working procedure for problems (or) Method of Mathematical Induction

- Suppose we wish to prove that a certain statement $S(n)$ is true for all integers $n \geq 1$. The method of providing such a statement on the basis of the Induction principle is called the method of mathematical induction. This method consists of the following two steps, respectively called the basis step and the induction step.

(1) Basis step:- Verify that the statement $S(1)$ is true that is, verify that ~~the~~ $S(n)$ is true for $n=1$.

(2) Induction step:- Assuming that $S(K)$ is true, where K is an integer ≥ 1, show that $S(K+1)$ is true

Solved problems

(1) By mathematical induction prove that

$$1^2+3^2+5^2+\cdots+(2n-1)^2=\frac{1}{3}n(2n-1)(2n+1)$$

Sol:- Let $S(n): 1^2+3^2+5^2+\cdots+(2n-1)^2=\frac{1}{3}n(2n-1)(2n+1)$

(i) Basis step:

Take $n=1$, we get $LHS=(2(1)-1)^2=(1)^2=1$

$RHS=\frac{1}{3}(1)(2(1)-1)(2(1)+1)=\frac{3}{3}=1$

$\therefore S(1)$ is true.

(ii) Induction step:

Assume that $S(n)$ is true for $n=K$, where $K\geq 1$.

i.e., $S(K): 1^2+3^2+5^2+\cdots+(2K-1)^2=\frac{1}{3}K(2K-1)(2K+1)$ —— (1)

Now, add the term $(2K+1)^2$ onto both sides of (1)

i.e., $1^2+3^2+5^2+\cdots+(2K-1)^2+(2K+1)^2=\frac{1}{3}K(2K-1)(2K+1)+(2K+1)^2$

$$=\frac{1}{3}(2K+1)\{K(2K-1)+3(2K+1)\}$$

$$=\frac{1}{3}(2K+1)\{2K^2-K+6K+3\}$$

$$=\frac{1}{3}(2K+1)\{2K^2+5K+3\}$$

$$=\frac{1}{3}(2K+1)(K+1)(2K+3)$$

i.e., $1^2+3^2+5^2+\cdots+(2K+1)^2 = \frac{1}{3}(K+1)(2K+1)(2K+3)$ — (2)

- Comparing (1) and (2), we conclude that $S(K+1)$ is true.
- Thus, the statement $S(K+1)$ is true whenever the statement $S(K)$ is true for $K\geq 1$.
- Hence, by mathematical induction, it follows that $S(n)$ is true for all integers $n\geq 1$.

(2) By mathematical induction prove that for any positive integer 'n' the number $11^{n+2}+12^{2n+1}$ is divisible by 133.

Sol:- Let $S_n: 11^{n+2}+12^{2n+1}$

(i) Basis step:

Take $n=1$, we get $S_1 = 11^{1+2}+12^{2+1} = 11^3+12^3 = 1331+1728 = 3059$.

Now, ~~check~~ $3059 = 23\times 133$, so that 133 divides 3059.

$\therefore$ S_n is true for $n=1$

(ii) Induction step:

Assume that S_n is true for $n=K$, where $K\geq 1$.

i.e., $S_K = 11^{K+2}+12^{2K+1}$ is divisible by 133 for $n=K\geq 1$

— (1)

Now, take $n = K+1$

i.e., $A_{K+1} = 11^{K+3} + 12^{2(K+1)+1}$

$= 11^{K+3} + 12^{2K+2+1}$

$= (11^{K+2} \times 11) + (12^{2K+1} \times 12^2)$

$= (11^{K+2} \times 11) + (12^{2K+1} \times 144)$

$= (11^{K+2} \times 11) + (12^{2K+1} \times (11+133))$

$= (11^{K+2} \times 11) + (12^{2K+1} \times 11) + (12^{2K+1} \times 133)$

$= (11^{K+2} + 12^{2K+1}) \times 11 + (12^{2K+1} \times 133)$

$= (S_K \times 11) + (12^{2K+1} \times 133)$

This representation shows that A_{K+1} is divisible by 133 when A_K is divisible by 133. This completes the proof of the required result by induction.

(3) Prove by mathematical Induction
$1\times3 + 2\times4 + 3\times5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$

Sol: Let $S(n): 1\times3 + 2\times4 + 3\times5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$

Take $n=1$, we get $LHS = (1)(1+2) = 3$

$$RHS = \frac{(1)(1+1)(2(1)+7)}{6} = \frac{18}{6} = 3$$

$\therefore S(1)$ is true

Assume that $S(K)$ is true, where $K \geq 1$
i.e., $S(K): 1\times3 + 2\times4 + \cdots + K(K+2) = \frac{K(K+1)(2K+7)}{6}$ is true — (1)

Now add $(K+1)(K+3)$ on both sides of (1)
i.e., $1\times3 + 2\times4 + \cdots + K(K+2) + (K+1)(K+3)$

$$= \frac{K(K+1)(2K+7)}{6} + (K+1)(K+3)$$

$$= \frac{K(K+1)(2K+7) + 6(K+1)(K+3)}{6}$$

$$= \frac{(K+1)\{2K^2+7K+6K+18\}}{6}$$

$$= \frac{(K+1)\{2K^2+13K+18\}}{6} = \frac{(K+1)(K+2)(2K+9)}{6}$$ — (2)

Comparing (1) and (2), we conclude that $S(K+1)$ is true when $S(K)$ is true. Hence proved.

4 prove by mathematical induction, for every positive integer n, 5 divides (n^5-n).

Sol: Let $S(n): (n^5-n)$ is divisible by 5

· $S(1): (1^5-1)=0$ is divisible by 5

$\therefore S(1)$ is true.

· Assume that $S(K)$ is true

i.e., $S(K): (K^5-K)$ is divisible by 5.

or $S(K): (K^5-K)=5a,\ a\in Z$ ——— 1

· Consider, $S(K+1)=(K+1)^5-(K+1)$

i.e., $S(K+1)=(K^5+5K^4+10K^3+10K^2+5K+1)-(K+1)$

using binomial expansi[on] $(K+1)^5$

$\Rightarrow S(K+1)=(K^5-K)+5(K^4+2K^3+2K^2+K)$

$\Rightarrow S(K+1)=5a+5(K^4+2K^3+2K^2+K)$

$\Rightarrow S(K+1)=5(a+K^4+2K^3+2K^2+K)=5b\text{(say)}$

$\Rightarrow S(K+1)=5b,\ b\in Z$ ——— 2

Comparing 1 and 2, we conclude that $S(K+1)$ is true

Thus by principle of mathematical induction $S(n)$ is true for all positive integers.

5 P.T $4n<(n^2-7)$ for all positive integers $n\geq 6$.

Sol:- Let $S(n): 4n<(n^2-7)$, $n\in Z^+$, $n\geq 6$

1 Basis step

Since $n\geq 6$, we shall take $n=6$ initially.

Hence, $S(6)=(4)(6)<(6^2-7)$ or $24<29$ is true.

$\therefore$ $S(6)$ is true.

2 Induction step

- Assume that $S(n)$ is true for $n=k$

i.e., $4k<(k^2-7), k\in Z^+, k\geq 6$ —— (1)

- Consider, $4(k+1)<((k+1)^2-7)$

i.e., $4(k+1)<(k^2+1+2k-7)$

i.e., $4(k+1)<\{(k^2-7)+(2k+1)\}$

i.e., $4k+4<(k^2-7)+(2k+1)$

- from (1), we have $4k<(k^2-7)$

~~also~~, Since $k\geq 6$, $2k+1=2(6)+1=7>4$ or $4<7$

i.e, $4<(2k+1)$ is valid.

Hence, $4(k+1)<(k+1)^2-7$ is true —— (2)

Comparing (1) and (2), we conclude that $S(k+1)$ is true.

Thus by the principle of mathematical induction $S(n)$ is true for all $n\geq 6$.

Exercise/ practice Questions

1. prove that every positive integer $n \geq 24$ can be written as sum of 5's and/or 7's.

2. prove by mathematical induction for every positive integer 8 divides $5^n + 2 \cdot 3^{n-1} + 1$

3. prove the statement by mathematical induction
$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

4. prove by mathematical induction
$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

5. prove by mathematical induction
$$\sum_{i=1}^{n} i(2^i) = 2 + (n-1)2^{n+1}$$

6. By Mathematical Induction P.T $(n!) \geq 2^{n-1}$, for all integers $n \geq 1$

7. ~~Prove that every positive integer n≥24 can be written as a sum of 5's and/or 7's~~

(*) Recursive Definitions

- An ordered set of real numbers $a_1, a_2, a_3, \ldots, a_n$ is called a sequence and it is usually denoted by $\{a_n\}$. a_n is called the n^{th} term or the general term of the sequence. We take a note that the general term of the sequence (Arithematic progression)
  $a, a+d, a+2d, \ldots$ is $a+(n-1)d$.

- Examples
  ① $1, 3, 5, \ldots$ represents the sequence $\{2n-1\}$
  ② $1, -1, 1, -1, \ldots$ represents the sequence $\{(-1)^{n-1}\}$
  ③ $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ represents the sequence $\left\{\frac{1}{n}\right\}$

- A sequence is represented by two methods.
  (i) Explicit method (ii) Recursive method.

- In the Explicit method the general term of the sequence is explicitly given. The various terms of the sequence are obtained by giving values for $n \in \mathbb{Z}^+$.

- In the Recursive method the general term relates to a few of its preceding terms.

Solved problems

1. Obtain a recursive definition for the sequence $\{a_n\}$ in each of the following.

(i) $a_n = 5n$ (ii) $a_n = 2 - (-1)^n$

Sol:- (i) $a_n = 5n$

$\therefore a_1 = 5,\ a_2 = 10,\ a_3 = 15,\ a_4 = 20, \ldots$

or $a_1 = 5,\ a_2 = 5+5,\ a_3 = 10+5,\ a_4 = 15+5, \ldots$

or $a_1 = 5,\ a_2 = a_1 + 5,\ a_3 = a_2 + 5,\ a_4 = a_3 + 5, \ldots$

Thus $a_n = a_{n-1} + 5$ where $a_1 = 5$ and $n > 1, n \in Z^+$

(ii) $a_n = 2 - (-1)^n$

We have $(-1)^n = \begin{cases} -1 & \text{if } n \text{ is an odd integer} \\ +1 & \text{if } n \text{ is an even integer} \end{cases}$

$\therefore a_1 = 3,\ a_2 = 1,\ a_3 = 3,\ a_4 = 1, \ldots$

or $a_1 = 3,\ a_2 = a_1 - 2,\ a_3 = a_2 + 2,\ a_4 = a_3 - 2, \ldots$

or $a_1 = 3,\ a_2 = a_1 + (2)(-1)^1,\ a_3 = a_2 + (2)(-1)^2,\ a_4 = a_3 + (2)(-1)^3$

That is $a_1 = 3,\ a_n = a_{n-1} + (2)(-1)^{n-1}$

Thus $a_n = a_{n-1} + (2)(-1)^{n-1}$ where $a_1 = 3$ and $n > 1, n \in Z^+$

(2) Find an explicit formula for $a_n = a_{n-1} + n$, $a_1 = 4$ for $n \geq 2$.

Sol:- Consider, $a_n = a_{n-1} + n$

$$= [a_{n-2} + (n-1)] + n$$

$$= [a_{n-3} + (n-2)] + (n-1) + n$$

$$= [a_{n-4} + (n-3)] + (n-2) + (n-1) + n$$

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$$= a_1 + 2 + 3 + 4 + \cdots + n$$

$$= a_1 - 1 + 1 + 2 + 3 + 4 + \cdots + n$$

i.e., $a_n = (a_1 - 1) + (1 + 2 + 3 + 4 + \cdots + n)$

Using $a_1 = 4$, we get

$$a_n = 3 + (1 + 2 + 3 + \cdots + n)$$

i.e., $a_n = 3 + \sum_{i=1}^{n} i$ But $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\therefore$ $\boxed{a_n = 3 + \frac{n(n+1)}{2}}$ is the required explicit formula.

(*) Fibonacci numbers

- The recursive definition is given by.

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2 \text{ and } F_0 = 0, F_1 = 1$$

- We shall find out a few terms of the sequence $\{F_n\}, n \geq 2$

$F_2 = F_1 + F_0 = 1+0 = 1$

$F_3 = F_2 + F_1 = 1+1 = 2$

$F_4 = F_3 + F_2 = 2+1 = 3$

$F_5 = F_4 + F_3 = 3+2 = 5$

$F_6 = F_5 + F_4 = 5+3 = 8$

$F_7 = F_6 + F_5 = 8+5 = 13$

$F_8 = F_7 + F_6 = 13+8 = 21$

$F_9 = F_8 + F_7 = 21+13 = 34$

The Fibonacci sequence $\{F_n\}$ is $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

(*) Lucas numbers

- The recursive definition is given by.

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2 \text{ and } L_0 = 2, L_1 = 1$$

- We shall find out a few terms of the sequence $\{L_n\}, n \geq 2$

$L_2 = L_1 + L_0 = 1+2 = 3$

$L_3 = L_2 + L_1 = 1+1 = 4$

$L_4 = L_3 + L_2 = 2+1 = 7$

$L_5 = L_4 + L_3 = 3+2 = 11$

$L_6 = L_5 + L_4 = 11+7 = 18$

$L_7 = L_6 + L_5 = 18+11 = 29$

$L_8 = L_7 + L_6 = 29+18 = 47$

$L_9 = L_8 + L_7 = 47+29 = 76$

The Lucas sequence $\{L_n\}$ is $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$

3 P.T for any positive integer n,

$$\sum_{i=1}^{n} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{n+2}}{2^n}, \quad F_n \text{ denote Fibonacci number.}$$

S:- Let, $\sum_{i=1}^{n} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{n+2}}{2^n}$, we have $F_n = F_{n-1} + F_{n-2}, n \geq 2$ and $F_0 = 0, F_1 = 1$

Step-1:- For $n=1$, LHS $= \frac{F_0}{2^1} = \frac{0}{2} = 0$

$$RHS = 1 - \frac{F_3}{2} = 1 - \frac{2}{2} = (1-1) = 0$$

$\therefore$ The result is true when $n=1$.

Step 2:- Let us assume that the result ~~to~~ be true for $n=k$

i.e., $\sum_{i=1}^{k} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{k+2}}{2^k}$ ——— (1)

Further, $\sum_{i=1}^{k+1} \frac{F_{i-1}}{2^i} = \frac{F_0}{2^1} + \frac{F_1}{2^2} + \frac{F_2}{2^3} + \cdots + \frac{F_{k-1}}{2^k} + \frac{F_{(k+1)-1}}{2^{k+1}}$

$$= \sum_{i=1}^{k} \frac{F_{i-1}}{2^i} + \frac{F_{(k+1)-1}}{2^{k+1}}$$

$$= \left[1 - \frac{F_{k+2}}{2^k}\right] + \frac{F_k}{2^{k+1}}, \text{ by using (1)}$$

$$= 1 - \frac{1}{2^{k+1}}\left[2F_{k+2} - F_k\right]$$

$$= 1 - \frac{1}{2^{k+1}}\left[(F_{k+2} - F_k) + F_{k+2}\right]$$

But $F_n = F_{n-1} + F_{n-2}$ and hence $F_{K+2} = F_{K+1} + F_K$.

or $F_{K+2} - F_K = F_{K+1}$

we use this result in the RHS.

$$\therefore \sum_{i=1}^{K+1} \frac{F_{i-1}}{2^i} = 1 - \frac{1}{2^{K+1}}\left[F_{K+1} + F_{K+2}\right] = 1 - \frac{1}{2^{K+1}} F_{K+3}$$

$$\text{i.e.,} \sum_{i=1}^{K+1} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{(K+1)+2}}{2^{K+1}} \quad \text{—— (2)}$$

- By comparing (1) and (2), we conclude that the result is true for $n = K+1$.
- Thus by the principle of mathematical induction the result is true for all positive integer values of n.

(4) For the Fibonacci sequence $F_0, F_1, F_2, \ldots$ prove that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$(\text{anything})^0 = 1$

Sol:- Let, $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

Step-1:- when $n=0$, $F_0 = \frac{1}{\sqrt{5}}\left[(1) - (1)\right] = 0$

when $n=1$, $F_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right] = 1$

But, we have $F_0 = 0$ and $F_1 = 1$ from Fibonacci recursive definition.

$\therefore$ The result is true when $n=0$ and 1.

Step 2:-
Assume that F_n is true for $n = 0, 1, 2, \ldots K$, where $K \geq 1$.

i.e., $F_K = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^K - \left(\frac{1-\sqrt{5}}{2}\right)^K\right]$ —— (1)

Consider, $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ and $F_0 = 0$, $F_1 = 1$ (by definition) —— (2)

Substitute $n = K+1$ in (2), gives

$$F_{K+1} = F_K + F_{K-1}$$

(17) $$F_{K+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{K} - \left(\frac{1-\sqrt{5}}{2}\right)^{K}\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{K-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{K-1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{K} - \left(\frac{1-\sqrt{5}}{2}\right)^{K} + \left(\frac{1+\sqrt{5}}{2}\right)^{K-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{K-1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{K-1}\left\{\left(\frac{1+\sqrt{5}}{2}\right)+1\right\} - \left(\frac{1-\sqrt{5}}{2}\right)^{K-1}\left\{\left(\frac{1-\sqrt{5}}{2}\right)+1\right\}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{K-1}\left\{\frac{3+\sqrt{5}}{2}\right\} - \left(\frac{1-\sqrt{5}}{2}\right)^{K-1}\left\{\frac{3-\sqrt{5}}{2}\right\}\right]$$

But, $(1+\sqrt{5})^2 = 1+2\sqrt{5}+5 = 6+2\sqrt{5} = 2(3+\sqrt{5})$

$(1-\sqrt{5})^2 = 1-2\sqrt{5}+5 = 6-2\sqrt{5} = 2(3-\sqrt{5})$

$\therefore \frac{3+\sqrt{5}}{2} = \frac{(1+\sqrt{5})^2}{2^2}$ and $\frac{3-\sqrt{5}}{2} = \frac{(1-\sqrt{5})^2}{2^2}$

Hence,

$$F_{K+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{K-1}\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{K-1}\left(\frac{1-\sqrt{5}}{2}\right)^{2}\right]$$

$$F_{K+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{K+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{K+1}\right] \quad \text{—— (2)}$$

- By comparing (1) and (2), we conclude that the result is true for $n = K+1$.
- Thus by the principle of mathematical induction the result is true for all positive integral values of n.

(5) If $F_0, F_1, F_2, \ldots$ are Fibonacci numbers, prove that $\sum_{i=0}^{n} F_i^2 = F_n \times F_{n+1}$

Sol: Let, $\sum_{i=0}^{n} F_i^2 = F_n \times F_{n+1}$

We have, $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ and $F_0 = 0, F_1 = 1$

Step-1:- When $n=0$, $LHS = F_0^2 = (0)^2 = 0$

$RHS = F_0 \times F_1 = (0) \times (1) = 0$

When $n=1$, $LHS = F_0^2 + F_1^2 = (0)^2 + (1)^2 = 1$

$RHS = F_1 \times F_2 = (1) \times (1) = 1$

$\therefore$ The result is true when $n=0$ and $n=1$.

Step 2:- Assume that the result is true for $n=K$.

i.e., $\sum_{i=0}^{K} F_i^2 = F_K \times F_{K+1}$ ——— (1)

Further, $\sum_{i=0}^{K+1} F_i^2 = F_0^2 + F_1^2 + F_2^2 + \cdots + F_K^2 + F_{K+1}^2$

i.e., $\sum_{i=0}^{K+1} F_i^2 = \sum_{i=0}^{K} F_i^2 + F_{K+1}^2$

i.e., $\sum_{i=0}^{K+1} F_i^2 = (F_K \times F_{K+1}) + F_{K+1}^2$

$$\sum_{i=0}^{K+1} F_i^2 = F_{K+1}(F_K + F_{K+1})$$

But $F_n = F_{n-1} + F_{n-2}$, this will give us $F_{K+2} = F_{K+1} + F_K$

$$\therefore \sum_{i=0}^{K+1} F_i^2 = F_{K+1}(F_{K+2})$$

or $$\sum_{i=0}^{K+1} F_i^2 = F_{K+1} \times F_{(K+1)+1} \quad \text{---} \; (2)$$

- By comparing (1) and (2), we conclude that the result is true for $n = K+1$.
- Thus by the principle of mathematical induction the result is true for all positive integer values of $n \geq 1$.

6. Let $a_0 = 1$, $a_1 = 2$, $a_2 = 3$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 3$. Prove that $a_n \leq 3^n$ $\forall n \in Z^+$

Sol:- Consider, $S(n): a_n \leq 3^n$

Step-1: $a_0 = 1 \leq 3^0$; $a_1 = 2 < 3^1$; $a_2 = 3 < 3^2 = 9$

$\therefore$ $S(0)$, $S(1)$, $S(2)$ are all true.

Step 2:- We shall assume that $S(n)$ is true for $n = k$,

where $n = 0, 1, 2, 3, \ldots, k$, $k \geq 2$

i.e., $\quad S_k = a_k \leq 3^k \quad$ —— ①

Now, $a_{k+1} = a_k + a_{k-1} + a_{k-2}$, using the definition of a_n

~~We note that $a_k \leq 3^k$ and $3^{k-1} \leq 3^k$, $3^{k-2} \leq 3^k$ is also~~

i.e., $a_{k+1} \leq 3^k + 3^{k-1} + 3^{k-2}$

$\leq 3^k + 3^k + 3^k$ because $3^{k-1} \leq 3^k$ and $3^{k-2} \leq 3^k$

$a_{k+1} = 3 \times 3^k = 3^{k+1}$ —— ②

Thus, $S(k+1)$ is true

Comparing ① and ②, we conclude that $S(k+1)$ is true

Thus by the principle of mathematical induction $S(n)$ is true for all positive integers.

Exercise / practice questions

1. The Fibonacci numbers are defined recursively by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Evaluate F_2 to F_{10}.

2. The Lucas numbers are defined recursively by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Evaluate L_2 to L_{10}.

3. If $L_0, L_1, L_2, \ldots$ are Lucas numbers, P.T.

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

4. Find an explicit definition of the sequence defined recursively by $a_1 = 7$, $a_n = 2a_{n-1} + 1$ for $n \geq 2$.

(*) Fundamental principles of counting

- The word 'Count' is nothing but finding the total number. It is an act of counting. Techniques of counting is essential both in Mathematics / Discrete Mathematics and computer science.
- There are two basic rules of counting

① Rule - 1: Sum Rule

- Let A_1 and A_2 be two events, where the event A_1 can happen in m_1 ways and event A_2 can happen in m_2 ways. Further the events A_1, A_2 cannot happen at the same time. Then either of the events (A_1 or A_2) can happen in (m_1+m_2) ways.
- In general, if the events $A_1, A_2, A_3, ---, A_n$ can happen respectively in $m_1, m_2, m_3, ---, m_n$ ways, such that no two events happen simultaneously. than the number of ways of the happening of one of the events (A_1 or A_2 or A_3 --- or A_n) is $m_1+m_2+m_3+---+m_n$.

Examples

① Suppose there are 6 pens and 4 pencils for writing, the number of ways of selecting a pen or a pencil for writing is $6+4=10$ ways.

(2) A college library has 40 textbooks on sociology and 50 textbooks dealing with anthropology. By the rule of sum, a student at this college can select among $40+50=90$ textbooks in order to learn more about of these two subjects.

(2) Rule - 2: Product Rule

- Let A_1 and A_2 be two events which happens one after another. If the event A_1 happens in m_1 ways and for each of these A_2 happens in m_2 ways, then both the events happens in $m_1 \cdot m_2$ ways.
- In general, if the events $A_1, A_2, A_3, \ldots A_n$ are events such that A_1 happens in m_1 ways, A_2 happens in m_2 ways, ..., A_n happens in m_n ways. Then the event A_1 followed by A_2, followed by A_3 ---- followed by A_n is a sequential order happens in $m_1 \cdot m_2 \cdot m_3 \cdots m_n$ ways.

Examples

(1) A TV news channel has 3 gents and 2 ladies as news readers to telecast a news bulletin with a combination of both. The number of possible ways is $3 \times 2 = 6$.

(2) A customer wants to change the ATM pin which as four digits. The number of ways it can happens with the repetition of digits is $10 \times 10 \times 10 \times 10 = 10000$, 10 being the number of digits 0 to 9. (possible no's 0000 to 9999 = 10,000)

Also the number of ways without the repetition of digits (all four digits are different) is $10 \times 9 \times 8 \times 7 = 5040$.

(3) An internet banking ~~customer~~ customer wants to set a password comprising of 6 English alphabets followed by 2 digits number.

- The number of possible ways with repetition is given by

$(26 \times 26 \times 26 \times 26 \times 26 \times 26) \times (10 \times 10) = (26)^6 \times (10)^2$

(There are 26 English alphabets and 10 number of digits).

- The number of possible ways without repetition is

$(26 \times 25 \times 24 \times 23 \times 22 \times 21) \times (10 \times 9)$.

(*) Factorial n, n being a natural number

- The product of all the first n natural numbers, that is from 1 to n is called "n factorial or factorial n". It is denoted by $\underline{n|}$ or $n!$

$\therefore n! = 1 \times 2 \times 3 \times \cdots \times n$

or

$n! = n \times (n-1) \times (n-2) \times \cdots 3 \times 2 \times 1$

Further, we have evidently

$n! = n \times (n-1)! = n(n-1)(n-2)!$ etc.

(*) Expression for ${}^nP_r$

- Let us suppose that we have 'n' different objects : $x_1, x_2, x_3, \cdots x_n$.
- Permutation being arrangement of objects, the first place can be filled up in 'n' ways and the number of objects left after this is $(n-1)$. The Second ~~object~~ place can be filled up in $(n-1)$ ways and the number of objects left after there two fillings is $(n-2)$. Similarly the third place can be filled up in $(n-2)$ ways.
- Continuing like this, the number of ways of filling the r^{th} place, where $r \leq n$ is filling up the first, second, third, $\cdots$ r^{th} places together is

$n(n-1)(n-2)\,---\,[n-(r-1)]$ ~~(1)~~

~~i.e., n(n-1)(n-2)~~

i.e., $\boxed{{}^nP_r \text{ or } P(n,r) = n(n-1)(n-2)\,---\,[n-(r-1)]}$ —— (1)

$$\boxed{{}^nP_r = P(n,r) = \frac{n!}{(n-r)!} \text{ (without repetition)}} \quad \text{—— (2)}$$

Note:- $\boxed{{}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!} \Rightarrow \boxed{{}^nP_n = n!}$ —— (3)

(*) Expression for ${}^nC_r$, where $n \geq r$

- Every combination is comprised of r different objects which can themselves be arranged in $r!$ ways.
- For a single combination of r different objects there are $r!$ arrangements. Therefore, for ${}^nC_r$ combination, the number of arrangements is obviously ${}^nC_r \cdot r!$, which is equal to the number of permutation of n things taken r at a time represented by ${}^nP_r$
- Equivalently, ${}^nC_r \cdot r! = {}^nP_r$
- using (2) in the RHS, we have

$$ {}^nC_r \cdot r! = \frac{n!}{(n-r)!} $$

i.e., $\boxed{{}^nC_r = \frac{n!}{(n-r)!\,r!}}$ —— (4)

Note :- $\boxed{{}^nC_{n-r} = \dfrac{n!}{(n-(n-r))(n-r)!} = \dfrac{n!}{(n-r)!\,r!} = {}^nC_r}$ —— (5)

- It is Convenient to use the following form of expression in the case of numerical Computations.

$$\boxed{{}^nP_r = n(n-1)(n-2)\,----\,(n-r-1)}$$

$$\boxed{{}^nC_r = \frac{{}^nP_r}{r!} = \frac{n(n-1)(n-2)---(n-r-1)}{r!}}$$

$$\boxed{{}^nC_0 = 1}$$

- Permutation with alike things/objects

- The number of permutation of n things taken all at a time is ${}^nP_n = n!$
- Suppose that out of 'n' things, n_1 things is of one type, n_2 things is of second type, --- n_r things is of r^{th} type where $n_1+n_2+---+n_r = n$ then the number of permulation of n things by taking all the things at a time is given by

$$\frac{n!}{n_1!\,n_2!\,----\,n_r!}$$

WORKED PROBLEMS

1 Find the value of n in the following cases:

(i) ${}^nP_2 = 132$ (ii) ${}^nP_3 = 3.{}^nP_2$ (iii) $2P(n,2) + 50 = P(2n,2)$

Sol: (i) ${}^nP_2 = 132$

i.e., $n(n-1) = 132$ or $n^2 - n = 132$

i.e., $n^2 - n - 132 = 0$

i.e., $(n-12)(n+11) = 0 \Rightarrow \boxed{n = 12}$ and $\boxed{n = -11}$

Thus the required $n = 12$

(ii) ${}^nP_3 = 3.{}^nP_2$

i.e., $n(n-1)(n-2) = 3n(n-1)$

i.e., $n - 2 = 3 \Rightarrow \boxed{n = 5}$

Thus the required $n = 5$.

(iii) $2P(n,2) + 50 = P(2n,2)$

i.e., $2n(n-1) + 50 = 2n(2n-1)$

i.e., $n(n-1) + 25 = n(2n-1)$

$n^2 - n + 25 = 2n^2 - n$

$n^2 = 25 \Rightarrow n = \pm 5.$

Thus the required $n = 5$.

(2) How many words can be formed by using the letters of the word 'CHEMISTRY'. Also, how many words of four distinct letters can be formed?

Sol:- The word CHEMISTRY has 9 distinct letters.

The number of words by taking all the letters at a time is ${}^9P_9$.

i.e., ${}^nP_r = n! = n(n-1)(n-2) --- (n-(r-1))$

$\Rightarrow {}^9P_9 = 9! = 362880$

The number of words having 4 distinct letters is ${}^9P_4$

i.e., ${}^9P_4 = 9 \times 8 \times 7 \times 6 = 3024$

(3) How many positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000?

Sol:- Here n must be of the form

$$n = x_1 x_2 x_3 x_4 x_5 x_6 x_7$$

where $x_1, x_2, x_3, ---, x_7$ are the given digits. ~~suppose we take $x_1 = 5$.~~ Then with $x_1 = 5, 6$ or 7.

3. Find the number of permutations of the letter of the following word.

(i) PROGRESS (ii) MATHEMATICS

(iii) ENGINEERING

Sol:- (i) PROGRESS word has 8 letters ($n=8$)

It has, R:2, S:2, P, O, G, E:1, numbers.

That is $n_1=2$, $n_2=2$, $n_3=1$, $n_4=1$, $n_5=1$, $n_6=1$

$n_1+n_2+n_3+n_4+n_5+n_6=8$ (Sum Rule)

∴ The required number of permutations is given by

$$\frac{8!}{2!\,2!\,(1!)^4} = \frac{40320}{4} = 10080$$

(ii) MATHEMATICS word has 11 letters ($n=11$)

It has M:2, A:2, T:2, H, E, S, I, C:1

∴ That is $n_1=2$, $n_2=2$, $n_3=2$, $n_4=n_5=n_6=n_7=n_8=1$

$n_1+n_2+n_3+n_4+n_5+n_6+n_7+n_8=11$

∴ The required number of permutation is given by

$$\frac{11!}{2!\,2!\,2!\,(1!)^5} = \frac{39916800}{8} = 4989600$$

4. How many positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000?

Sol: Given, the set digits 3, 4, 4, 5, 5, 6, 7

- Number of digits = 7
- To make a positive integer that exceeds 5,000,000 we need to keep 5, 6 or 7 in the 1st position of the resultant number
- Let $n = x_1x_2x_3x_4x_5x_6x_7$, where $x_1, x_2, \ldots, x_7$ are the given digits.
- Suppose we take $x_1 = 5$. Then $x_2x_3x_4x_5x_6x_7$ is an arrangement of the remaining 6 digits which contains two 4's and one each of 3, ~~5~~ 6, 7.

$\therefore$ The no of such arrangments $= \dfrac{6!}{2!\,1!\,1!\,1!\,1!} = 360$

- Suppose we take $x_1 = 6$. Then $x_2x_3x_4x_5x_6x_7$ is an arrangement of 6 digits which contains two each of 4 and 5 and one each of 3 and 7.

$\therefore$ The no of such arrangements $= \dfrac{6!}{1!\,2!\,2!\,1!} = 180.$

- Similarly, if we take $x_1 = 7$, the number of arrangements is $\dfrac{6!}{1!\,2!\,2!\,1!} = 180.$

Accordingly, by the sum Rule, the number of n's of the desired type is

$$360 + 180 + 180 = 720.$$

(5) A certain question paper contains two parts A and B, each containing 4 questions. How many different ways a student can answer 5 questions by selecting atleast 2 questions from each part?

Sol:- Given, Part A – 4 questions
Part B – 4 questions

Part A	Part B
Q1	Q1
Q2	Q2
Q3	Q3
Q4	Q4

- Atleast 2 questions to be selected from each part to fulfill the requirement of 5 questions.
- The options are as follows.
- (Part-A – 2Q and Part B – 3Q)

or

(Part A – 3Q and Part B – 2Q)

i.e, $^4C_2 \times {}^4C_3$; $^4C_3 \times {}^4C_2$

i.e, $^4C_2 \times {}^4C_3 = \frac{4!}{2!(4-2)!} \times \frac{4!}{3!(4-3)!} = \frac{4!}{2!2!} \times \frac{4!}{3!1!} = \frac{144}{6} = 24$ ways

Each being 24 ways.

The required number of ways is equal to

$$24 + 24 = 48 \text{ ways.}$$

6. How many numbers greater than 1000000 can be formed by using the digits 1, 2, 2, 2, 4, 4, 0?

Sol: The number of digits are 7 in number.

Let $n = d_1 d_2 d_3 d_4 d_5 d_6 d_7$, d_i's being digits. If n has to greater than 1000000 which also has 7 digits, it is necessary that the number has to begin with 1 or 2 or 4.

- Case (i) :- Suppose $d_1 = 1$, the rest of the 6 digits has 3 Nos. of 2's, 2 Nos. of 4's, 1 No. of 0

$\therefore$ The number of such permutations $= \dfrac{6!}{3!\,2!} = 60$

- Case (ii) :- Suppose $d_1 = 2$, the rest of the 6 digits has 2 Nos. of 2, 2 Nos. 4's, 1 each of 1 and 0

$\therefore$ The number of such permutations $= \dfrac{6!}{2!\,2!} = 180$

- Case (iii) :- Suppose $d_1 = 4$, the rest of the 6 digits has 3 Nos of 2's, 1 each of 1, 4 and 0

$\therefore$ The number of such permutations $= \dfrac{6!}{3!\,1!} = 120$

By applying the sum rule we have $60 + 180 + 120 = 360$

Thus the required numbers greater than 1000000 is 360.

7 A certain question paper has 3 parts A, B, C with four questions in part A, Five in B and six in C. It is required to answer seven questions by selecting at least two from each part. In how many different ways student can answer seven questions.

Sol:- Given, Part A – 4 questions

Part B – 5 questions

Part C – 6 questions

- At least 2 questions to be selected from each part to fulfill the requirement of 7 questions.
- The options are as follows

(i) 2 questions from part A, 2 from part B and 3 from C.

(ii) 2 questions from part A, 3 from part B and 2 from C.

(iii) 3 questions from part A, 2 from part B and 2 from C.

(i) no of selection = ${}^4C_2 \times {}^5C_2 \times {}^6C_3$ = 1200 ways

(ii) no of selection = ${}^4C_2 \times {}^5C_3 \times {}^6C_2$ = 900 ways

(iii) no of selection = ${}^4C_3 \times {}^5C_2 \times {}^6C_2$ = 600 ways

∴ Total no of possible selections = 1200 + 900 + 600 = 2700

Exercise (practice Questions)

1. A total amount of Rs. 1500 is to be distributed to three students A, B, C. In how many ways distribution can be done in the multiple of Rs. 100 if
   (i) Every students gets at least Rs. 300
   (ii) A must get at least Rs. 500, B and C must get at least Rs. 400 each.

2. Find the number of ways of arrangement of the letters of the word 'TALLAHASSEE' which have no adjacent A's.

3. In how many ways one can distribute 8 identical marbles in 4 distinct containers so that (i) no container is empty (ii) the fourth container has an odd number of marbles in it.

4. Find the number of permutations of the letters of the word MASSASAUGA. In how many of these all four A's are together? How many of them begin with S?

(*) The Binomial Theorem

- If 'n' is a positive integer, we are already familiar with the expansion of $(x+y)^n$

i.e., $$\boxed{(x+y)^n = x^n + {}^nC_1 x^{n-1}y + {}^nC_2 x^{n-2}y^2 + \cdots + {}^nC_r x^{n-r}y^r + \cdots + y^n} \quad \text{--- (1)}$$

- There are $(n+1)$ terms in the expansion. The general term of the expansion is denoted by T_{r+1}.

i.e., $$\boxed{T_{r+1} = {}^nC_r x^{n-r} y^r = {}^nC_{n-r} x^r y^{n-r}} \quad \text{--- (2)}$$

- Hence (1) in the summation form is as follows.

$$\boxed{(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r = \sum_{r=0}^{n} {}^nC_r x^r y^{n-r}} \quad \text{--- (3)}$$

(*) Generalized Binomial Theorem or Multinomial Theorem

The generalized form of Binomial Theorem is as follows

$$\boxed{(x_1 + x_2 + \cdots + x_r)^n = \sum \binom{n}{n_1, n_2 \ldots n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}} \quad \text{--- (4)}$$

Solved problems

1. Find the following coefficients
   (i) $a^5 b^2$ in the expansion of $(2a-3b)^7$
   (ii) $x^6 y^3$ in the expansion of $(x+2y)^9$

S:- (i) We have $(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r$ —— (1)

$$\therefore \quad (2a-3b)^7 = \sum_{r=0}^{7} {}^7C_r (2a)^{7-r} (-3b)^r$$

$$\text{or} \quad (2a-3b)^7 = \sum_{r=0}^{7} {}^7C_r 2^{7-r} (-3)^r a^{7-r} b^r$$

By taking $r=2$, we have

$$(2a-3b)^7 = \sum_{r=2}^{7} {}^7C_2 2^5 (-3)^2 a^5 b^2$$

The Coefficient of $a^5 b^2 = {}^7C_2 2^5 (-3)^2$

$$= 21 \times 32 \times 9$$

$$= 6048$$

Thus the required Coefficient is given by 6048

(ii) Again from (1) we have,

$$(x+2y)^9 = \sum_{r=0}^{9} {}^9C_r x^{9-r} (2y)^r$$

By taking $r=3$, we have

$$(x+2y)^9 = \sum_{r=3}^{9} {}^9C_3 x^6 y^3 (2)^3$$

i.e., $(x+2y)^9 = \sum_{r=3}^{9} 2^3 \cdot {}^9C_3\, x^6 y^3$

The Coefficient of $x^6y^3 = 2^3\,{}^9C_3 = 8\times\left(\frac{9\times 8\times 7}{6}\right) = 672$

Thus the required Coefficient is given by 672.

(2) Determine the Coefficient of xyz^2 in the expansion of $(2x-y-z)^4$

Sol:- The general terms is given by

$$(x_1+x_2+\cdots+x_r)^n = \sum \binom{n}{n_1,\ n_2 \cdots n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

i.e., $(2x-y-z)^4 = \binom{4}{n_1\ n_2\ n_3}(2x)^{n_1}(-y)^{n_2}(-z)^{n_3}$

By taking $n_1=1,\ n_2=1,\ n_3=2$, we have

$$\binom{4}{1\ 1\ 2}(2x)^1(-y)^1(-z)^2 = \frac{4!}{1!\,1!\,2!}(-2xyz^2) = -24xyz^2$$

Thus the required Coefficient of xyz^2 is -24

3 Find the Coefficient of:

(i) x^9y^3 in the expansion of $(2x-3y)^{12}$

(ii) x^{12} in the expansion of $x^3(1-2x)^{10}$

Sol:- (i) We have, by the binomial theorem

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^r y^{n-r} \quad \text{——— (1)}$$

i.e., $$(2x-3y)^{12} = \sum_{r=0}^{12} {}^{12}C_r\,(2x)^r(-3y)^{12-r}$$

$$(2x-3y)^{12} = \sum_{r=0}^{12} 2^r(-3)^{12-r}x^r y^{12-r}\ {}^{12}C_r$$

By taking $r=9$, we have

$$(2x-3y)^{12} = \sum_{r=9}^{12} 2^9(-3)^3 x^9y^3\ {}^{12}C_r$$

The coefficient of $x^9y^3 = {}^{12}C_r\, 2^9(-3)^3$

$$= -(2^9\times 3^3)\times \frac{12!}{9!\,3!}$$

$$= -2^9\times 3^3\times \frac{12\times 11\times 10}{6}$$

$$= -2^{10}\times 3^3\times 11\times 10$$

ii) Consider,

$$x^3(1-2x)^{10} = \sum_{r=0}^{10} {}^{10}C_r (1)^r (-2x)^{10-r} \cdot x^3$$

$$= \sum_{r=0}^{10} {}^{10}C_r (1)^r (-2)^{10-r} x^{10-r} x^3$$

$$= \sum_{r=0}^{10} {}^{10}C_r (1)^r (-2)^{10} \cdot (-2)^{-r} x^{10+3-r}$$

$$x^3(1-2x)^{10} = \sum_{r=0}^{10} {}^{10}C_r (1)^r (1024)(-2)^{-r} x^{13-r}$$

By taking $r=1$ we have,

$$x^3(1-2x)^{10} = \sum_{r=1}^{10} {}^{10}C_1 (1)^1 (1024)(-2)^{-1} x^{12}$$

The coefficient of $x^{12} = {}^{10}C_1 (1)^1 (1024)\left(-\frac{1}{2}\right)$

$$= -5120$$

4. Find coefficient of

(i) x^0 in the expansion of $\left(3x^2 - \frac{2}{x}\right)^{15}$

(ii) $x^{11}y^4$ in the expansion of $(2x^3 - 3xy^2 + z^2)^6$

Sol:- (i) We have, by the binomial theorem

$$\left(3x^2 - \frac{2}{x}\right)^{15} = \sum_{r=0}^{15} \binom{15}{r} (3x^2)^r \left(-\frac{2}{x}\right)^{15-r}$$

$$= \sum_{r=0}^{15} \binom{15}{r} 3^r (-2)^{15-r} x^{2r} \left(\frac{1}{x}\right)^{15-r}$$

$$= \sum_{r=0}^{15} \binom{15}{r} 3^r (-2)^{15-r} x^{3r-15}$$

By taking $r=5$, we have

$$\left(3x^2 - \frac{2}{x}\right)^{15} = \sum_{r=5}^{15} \binom{15}{5} 3^5 (-2)^{10} x^0$$

The coefficient of $x^0 = \binom{15}{5} 3^5 (-2)^{10}$

$$= \frac{15!}{10!\,5!} \times 3^5 \times (-2)^{10}$$

(ii) By the Multinomial theorem, we have

$$(x_1+x_2+\dots+x_r)^n=\sum\binom{n}{n_1,n_2,\dots,n_r}x_1^{n_1}\cdot x_2^{n_2}\dots x_r^{n_r}$$

$\therefore$ The general term in the expansion of

$$(2x^3-3xy^2+z^2)^6=\sum\binom{6}{n_1\ n_2\ n_3}(2x^3)^{n_1}(-3xy^2)^{n_2}(z^2)^{n_3}$$

$$=\sum\binom{6}{n_1\ n_2\ n_3}2^{n_1}(-3)^{n_2}x^{3n_1}(xy^2)^{n_2}z^{2n_3}$$

$$=\sum\binom{6}{n_1\ n_2\ n_3}2^{n_1}(-3)^{n_2}x^{3n_1+n_2}y^{2n_2}z^{2n_3}$$

$\therefore$ For $n_3=0, n_2=2, n_1=3$ this becomes

$(2x^3-3xy^2+z^2)^6$

$$=\sum\binom{6}{3\ 2\ 0}(2^3)(-3)^2x^{11}y^4z^0$$

The Coefficient of $x^{11}y^4 = 2^3(-3)^2\binom{6}{3\ 2\ 0}$

$$=\frac{6!}{3!\,2!\,0!}\times 72$$

$$=4320$$

$x^{11}y^4z^0$

$z^0 \Rightarrow n_3=0$

$2n_2=4$

$\boxed{n_2=2}$

$3n_1+n_2=3n_1+2$

$3n_1+2=11$

$3n_1=11-2$

$\boxed{n_1=\frac{9}{3}=3}$

Exercise

1. Using the Binomial theorem find the Coefficient of x^5y^2 in the expansion of $(x+y)^7$

2. Find the Coefficient of $x^{11}y^4$ in the expansion of $(2x^3-3xy^2+z^2)^6$

3. Find the Coefficient of $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$.

4. ~~Find the sum of~~

4. ~~Determine the coefficients of~~

(*) Combinations with Repetitions

- The number of Combination of n distinct things taken r at a time with possible repetition is given by ${}^{n+r-1}C_r$.

- Note:- Using the property ${}^{n}C_r = {}^{n}C_{n-r}$

  we have, ${}^{(n+r-1)}C_r = {}^{(n+r-1)}C_{n-1}$

- Examples

Objects	Combination without repetition	Combination with repetition
a, b, c	Taken 2 at a time ($n=3$, $r=2$) ab, bc, ca = 3 ${}^{3}C_2 = 3$	Taken 2 at a time ($n=3$, $r=2$) ab, bc, ca aa, bb, cc $\}$ = 6 ${}^{(n+r-1)}C_r = {}^{4}C_2 = \frac{4\times 3}{1\times 2} = 6$
a, b, c, d	Taken 3 at a time ($n=4$, $r=3$) abc, abd, bcd, cda = 4 ${}^{4}C_3 = 4$	Taken 3 at a time ($n=4$, $r=3$) abc, abd, bcd, cda, aab, aac, aad, bba, bbc, bbd, cca, ccb, ccd, dda, ddb, ddc, aaa, bbb, ccc, ddd = 20 ${}^{(n+r-1)}C_r = {}^{6}C_3 = 20$

Solved problems

(1) A total amount of Rs. 1500 is to be distributed to three students A, B, C. In how many ways the distribution can be made in multiples of Rs. 100

(i) if every students gets at least Rs 300

(ii) if A must get at least Rs. 500, and B and C must get at least Rs. 400 each.

Sol: There are 15 objects (15 hundred Rupees notes) to be distributed among 3 students A, B, C.

(i) Every students gets atleast Rs. 300

- Distribute Rs. 300 to every students
- Remaining 6 notes should be distributed among 3 students.

i.e., $r=6$, $n=3$

- Number of ways of distribution $= {}^{n+r-1}C_r$

$$= {}^{3+6-1}C_6 = {}^{8}C_6 = 28 \text{ ways}$$

(ii) Distribute Rs 500 to A, Rs 400 to B and C each.

Remaining 2 notes of 100 should be distributed among 3 students A, B, C.

i.e., $r=2$, $n=3$

No of ways of distributing $= {}^{n+r-1}C_r = {}^{4}C_2 = \frac{4\times3}{2} = 6$ ways

2. In how many ways can 10 identical Coins be distributed among 5 children if
(i) There are no restrictions
(ii) Each child gets atleast one coin
(iii) The oldest child gets atleast 2 Coins.

Sol:- 10 identical coins is to be distributed among 5 children.

(i) We have $r = 10$, $n = 5$ and the required number of ways is given by

$$^{(n+r-1)}C_r = {}^{14}C_{10} = {}^{14}C_4 = \frac{14 \times 13 \times 12 \times 11}{24} = 1001.$$

Thus the required number of ways is 1001

(ii) Firstly, we have to distribute 1 Coin to each Child, with the result we are left with 5 coins for distribution among 5 children. Hence we have $r = 5$, $n = 5$ and the number of ways is

$$^{(n+r-1)}C_r = {}^{9}C_5 = {}^{9}C_4 = \frac{9 \times 8 \times 7 \times 6}{24} = 126.$$

Thus the required number of ways is 126

(iii) Firstly, we have to give 2 coins to the oldest child, with the result we are left with 8 coins for distribution among 5 children. Hence we have, $r=8$, $n=5$, and the number of ways

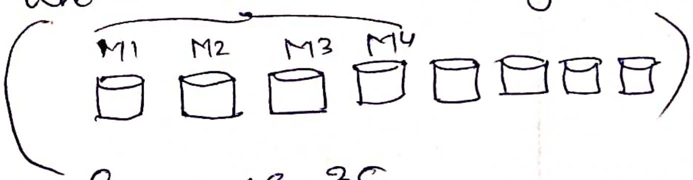
$$^{(n+r-1)}C_r = {}^{12}C_8 = {}^{12}C_4 = \frac{12\times11\times10\times9}{24} = 495.$$

Thus the required number of ways is 495.

(3) In how many ways one can distribute 8 identical marbles in 4 distinct containers so that (i) no container is empty
(ii) the fourth container has an odd number of marbles in it.

Sol: (i) We have to first put one marbel into each of the four containers, with the result we are left with four marbles. There have to distributed among 4 containers.

We have $n=4$, $r=4$ and the number of ways



$$^{(n+r-1)}C_r = {}^7C_4 = 35$$

Thus the required number of ways 35.

(ii) Fourth container can contain 1 or 3 or 5 or 7 marbles. This will give us 4 options

No of marbels in 4th container	Distributed among 3 containers	Number of ways $^{(n+r-1)}C_r$
1	7 marbels among 3 containers ($n=3$, $r=7$)	$^9C_7 = 36$
3	5 marbels among 3 containers ($n=3$, $r=5$)	$^7C_5 = 21$

5	3 markels among 3 Containers ($n=3$, $r=3$)	${}^5C_3 = 10$
7	1 mabel among 3 Containers ($n=3$, $r=1$)	${}^3C_1 = 3$

Thus the required number of ways

$36+21+10+3=70$